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AN ASSESSMENT OF THE SPECTRAL PROPERTIES OF THE MATRICES OBTAINED IN THE BOUNDARY ELEMENT METHODS

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***Abstract**. It is well known that the matrix **H** of cinematic transformations one arrives at in the conventional boundary element method is singular, for a finite domain, as a consequence of the fact that rigid body displacements do not produce deformations in an elastic body. A little more than a decade ago, the author introduced the "hybrid boundary element method", in which a symmetric, positive semi-definite flexibility matrix **F** is obtained on the basis of a sound variational principle. Along with this novel formulation, concepts of generalized inverse matrices had to be considered for the adequate understanding and manipulation of the matrix singularities involved. Motivated by the author's first accomplishments, De Figueiredo derived a few years later the "hybrid displacement boundary element method", a variational counterpart that gives rise to the same matrix **F**, but this time in relation to the matrix **G** of the conventional boundary element method. Quite recently, the author decided to investigate the spectral properties of this flexibility-like matrix **G**, in order do demonstrate that, adequately obtained, this is also a singular matrix (as only related to forces in balance). Such a singularity is a welcome feature one could and should take advantage of. The present paper is an attempt of assessing the spectral properties of the matrices **H**, **G** and **F**, according to the roles they play in the different boundary element formulations. It is demonstrated that such properties are interrelated in the formulations outlined. A not unremarkable consequence of this compared study is the conclusion that both the conventional and the hybrid displacement boundary element methods need some conceptual improvements in their formulations, in order to become completely consistent.*

***Key words**: Boundary element methods, variational methods, generalized inverse matrices.*

1. SOME BASIC CONSIDERATIONS ON THE FUNDAMENTAL SOLUTIONS

The developments carried out in items 1 and 2 are taken from Dumont (1998). Consider the fundamental solution of a generic three-dimensional elasticity problem, expressed in terms of displacements u_i^* measured at a given point for a given coordinate direction "*i*" of the domain, caused by some arbitrary, singular force p_m^* acting according to a given degree of freedom "*m*" (the index "*m*" characterizes both a point and a direction in the domain):

$$u_i^* = u_{im}^* p_m^* + u_{is}^r r_s \equiv (u_{im}^* + u_{is}^r C_{sm}) p_m^* \qquad (1)$$

This fundamental solution is usually given in the literature by the function u^*_{im} alone, implicitly related to unitary forces p^*_m. The complete representation of eq. (1) is both mathematically and physically more adequate, since it is stated for an arbitrary (not unitary) singular force p^*_m (in which "*" means "fundamental solution") and a term is added to take into account the arbitrary rigid body displacements, as characterized by the superscript "*r*". In the rigid body displacement functions u^r_{is}, "*s*" refers to the rigid body displacement being interpolated. The quantities r_s are arbitrary constants, which may be correlated to the singular forces p^*_m through some arbitrary matrix C_{sm} of constants. In this paper, subscripts "*m*" and "*n*" refer to degrees of freedom of discretized quantities; subscripts "*s*" and "*t*" refer to rigid body displacements; and subscripts "*i*" and "*j*" are related to the coordinate directions.

The stresses at a given point of the domain are obtained from eq. (1) as

$$\sigma^*_{ij} = \sigma^*_{ijm} p^*_m \quad \text{such that} \quad \sigma^*_{ij},_j = \sigma^*_{ijm},_j \, p^*_m = 0 \text{ in } \Omega \tag{2}$$

as a property of a fundamental solution. Moreover, in a vicinity Ω_0 of the point of application of the singular force p^*_m, $\int_{\Omega_0} \sigma^*_{ijm},_j \, d\Omega$ is equal to either 1 or zero, depending on whether the subscripts "*i*" and "*m*" refer to the same degree of freedom or not. From the stresses in eq. (2) one derives the traction forces along the boundary Γ as

$$t^*_i = p^*_{im} p^*_m \tag{3}$$

2. THE TRADITIONAL BOUNDARY ELEMENT EQUATION

The matrix equations of the traditional boundary element method may be stated, starting from minimum residual considerations and making use of eqs. (1) and (3), as

$$\begin{aligned} & p^*_m \left(\int_\Gamma p^*_{im} u_{in} d\Gamma - \int_\Omega \sigma^*_{ijm},_j \, u_{in} d\Omega \right) d_n = \\ & = p^*_m \left(\int_\Gamma u^*_{im} t_{in} d\Gamma \right) t_n + p^*_m \left(\int_\Gamma C_{sm} u^r_{is} t_{in} d\Gamma \right) t_n + p^*_m \left(\int_\Omega u^*_{im} b_i d\Omega \right) + p^*_m \left(\int_\Omega C_{sm} u^r_{is} b_i d\Omega \right) \end{aligned} \tag{4}$$

in which u_{in} and t_{in} are interpolation functions for displacements u_i, in terms of some nodal parameters d_n, and traction forces t_i, in terms of some nodal parameters t_n, respectively (usually $u_{in} \equiv t_{in}$). Body forces are taken into account by the vector b_i.

Considering that p^*_m is arbitrary, eq. (4) leads to the known matrix equation

$$\mathbf{Hd} = \mathbf{Gt} + \mathbf{b} \tag{5}$$

In equation above,

$$\mathbf{H} \equiv H_{mn} = \int_\Gamma p^*_{im} u_{in} d\Gamma - \int_\Omega \sigma^*_{ijm},_j \, u_{in} d\Omega \tag{6}$$

is given by the expression in the first brackets in eq. (4), supposing that the singularities of the boundary integral have been properly dealt with, also observing eq. (2). The matrix

$$\mathbf{G} \equiv G_{mn} = \int_\Gamma u^*_{im} t_{in} d\Gamma \tag{7}$$

is given by the boundary integral in the second brackets of eq. (4), an improper integral that may also present some quasi-singularities (Dumont, 1994).

The terms $\mathbf{d} \equiv d_n$ and $\mathbf{t} \equiv t_n$ in eq. (5) are vectors corresponding to boundary displacement and traction parameters, respectively.

Finally, one has in eq. (5) the vector **b** of nodal displacements equivalent to body forces:

$$\mathbf{b} \equiv b_m = \int_\Omega u^*_{im} b_i d\Omega \tag{8}$$

Equation (4) can only lead to eq. (5) if the terms related to the rigid body displacements u^r_{is} vanish, for arbitrary p^*_m and C_{sm}, that is, if

$$\int_\Gamma u^r_{is} t_{in} d\Gamma \; t_n + \int_\Omega u^r_{is} b_i d\Omega \;\equiv 0 \tag{9}$$

This equation means that the assumed traction forces along the boundary should be in equilibrium with the body forces as a premise. It seems that this fact has not been adequately dealt with in the literature.

2.1 Constructing a Spectrally Admissible Matrix G

Equation (12) may be represented in matrix notation as

$$\mathbf{R^T t} + \mathbf{b^r} = \mathbf{0} \tag{10}$$

in which

$$\mathbf{R} \equiv R_{ns} = \int_\Gamma u^r_{is} t_{in} d\Gamma \quad \text{and} \quad \mathbf{b}^r \equiv b_s = \int_\Omega u^r_{is} b_i d\Omega \tag{11}$$

are a rectangular matrix with as many columns as the number of rigid body displacements u^r_{is} and a vector of equivalent nodal displacements obtained in terms of the (mixed) virtual work done by the body forces on u^r_{is}, respectively.

Consider a rectangular matrix **Z**, the columns of which are an orthogonal basis of the columns of **R**, that is, such that $\mathbf{Z}^\mathrm{T}\mathbf{Z} = \mathbf{I}$ and $\left(\mathbf{Z}\,\mathbf{Z}^\mathrm{T}\right)\left(\mathbf{Z}\,\mathbf{Z}^\mathrm{T}\right) = \mathbf{Z}\,\mathbf{Z}^\mathrm{T}$. The idempotent matrix $\mathbf{Z}\,\mathbf{Z}^\mathrm{T}$ is the *orthogonal projector* on the space of the inadmissible, unbalanced traction force parameters **t** (Ben-Israel and Greville, 1980). For elasticity problems, the rigid body displacement functions u^r_{is} may be defined in infinite ways. However, the resulting idempotent matrix $\mathbf{Z}\,\mathbf{Z}^\mathrm{T}$ is unique. Then, it follows from the definition of **Z** that

$$\mathbf{R} = \mathbf{Z}\boldsymbol{\lambda} \tag{12}$$

in which $\boldsymbol{\lambda}$ is a non-singular square matrix readily obtained as

$$\boldsymbol{\lambda} = \mathbf{Z^T R} \tag{13}$$

If the traction force parameters **t** satisfy eq. (10), a condition for eq. (5) to be valid, it follows from eqs. (12) and (13) that

$$\mathbf{Z}^\mathrm{T}\mathbf{t} + \boldsymbol{\lambda}^{-\mathrm{T}}\mathbf{b}^r = \mathbf{0} \tag{14}$$

Pre-multiplying equation above by **Z** and subtracting **t** from both sides yields the condition that **t** must satisfy to ensure the validity of eq. (5):

$$\mathbf{t} = (\mathbf{I} - \mathbf{Z}\,\mathbf{Z}^\mathbf{T})\mathbf{t} - \mathbf{Z}\,\boldsymbol{\lambda}^{-\mathbf{T}}\mathbf{b}^r \tag{15}$$

If this relationship is valid, then eq. (5) should be re-written as

$$\mathbf{Hd} = \mathbf{G}(\mathbf{I} - \mathbf{Z}\,\mathbf{Z}^\mathbf{T})\,\mathbf{t} + (\mathbf{b} - \mathbf{G}\,\mathbf{Z}\boldsymbol{\lambda}^{-\mathbf{T}}\mathbf{b}^r) \qquad \text{or} \qquad \mathbf{Hd} = \mathbf{G}_a\,\mathbf{t} + \mathbf{b}_a \tag{16}$$

in which $\mathbf{G}_a \equiv \mathbf{G}(\mathbf{I} - \mathbf{Z}\mathbf{Z}^\mathbf{T})$ is the *admissible* part of the matrix $\mathbf{G}$, obtained through the orthogonal projection given by $\left(\mathbf{I}-\mathbf{Z}\,\mathbf{Z}^T\right)$, and $\mathbf{b}_a \equiv (\mathbf{b}-\mathbf{G}\mathbf{Z}\boldsymbol{\lambda}^{-\mathrm{T}}\mathbf{b}^r)$ is a vector of *admissible* nodal displacements related to the body forces. The admissible matrix $\mathbf{G}_a$, as defined in eq. (16), is singular. It is worth establishing that

$$\mathrm{Rank}(\mathbf{G}_a) = \mathrm{rank}\left(\mathbf{I}-\mathbf{Z}\,\mathbf{Z}^T\right) \tag{17}$$

a feature that can only be inferred physically. In fact, the matrix **G** is a flexibility-type transformation matrix, which must always yield some non-trivial equivalent nodal displacement vector to any set of traction force parameters **t**, if one is dealing with an elastic body. Then, owing to this physical property, **G** should be non-singular. However, depending on the set of rigid body displacement functions u_{is}^r that appears in the definition of the fundamental solution, as given in eq. (1), **G** may become singular or ill conditioned. Regardless the conditioning of matrix **G**, the rank of matrix $\mathbf{G}_a$ is always well defined according to eq. (17), since $\mathbf{G}_a$ is by construction independent of the rigid body displacement functions u_{is}^r. The conventional boundary element formulation relies on the fact that the matrix **G** is non-singular and hopefully not ill conditioned. All considerations of the present paper are based on the effectively reliable premise expressed by eq. (17).

One might attempt to solve eq. (16) for the admissible traction parameters $\mathbf{t}$:

$$\mathbf{t} = \mathbf{G}_a^{(-1)}(\mathbf{H}\mathbf{d}-\mathbf{b}_a) \tag{18}$$

An apparent difficulty in obtaining eq. (18) lies in the fact that $\mathbf{G}_a$ is singular. Fortunately, equation system (16) corresponds mathematically to a problem proposed and solved by Bott and Duffin in1953 (apud Ben-Israel and Greville, 1980). According to that, one proposes following restricted inverse for $\mathbf{G}_a$:

$$\mathbf{G}_a^{(-1)} = (\mathbf{I}-\mathbf{Z}\,\mathbf{Z}^T)(\mathbf{G}_a+\mathbf{Z}\,\boldsymbol{\gamma}\,\mathbf{Z}^T)^{-1} \tag{19}$$

which is more adequate than the Bott-Duffin inverse, since it contains a symmetric, positive definite, but otherwise arbitrary, matrix $\boldsymbol{\gamma}$, which may be chosen in order to ensure that the elements of $\mathbf{Z}\,\boldsymbol{\gamma}\,\mathbf{Z}^T$ and $\mathbf{G}_a$ have approximately the same magnitude, thus avoiding round-off errors during the numerical computations. In elastostatics, for instance, the elements of the matrix **G** are inversely proportional to the shear modulus, which does not affect the orthogonal basis **Z**. Since $\mathbf{G}_a$ and $\mathbf{Z}\,\boldsymbol{\gamma}\,\mathbf{Z}^T$ are complementary matrices $(\mathbf{G}_a\mathbf{Z}\,\boldsymbol{\lambda}\,\mathbf{Z}^T \equiv \mathbf{0})$, $\mathbf{G}_a+\mathbf{Z}\,\boldsymbol{\gamma}\,\mathbf{Z}^T$ is always well conditioned (see eq. (17) and subsequent considerations).

Alternatively, one could think in obtaining the matrix $\mathbf{C} \equiv C_{sm}$, in eq. (4), in such a way that, in absence of body forces, the nodal displacements equivalent to any set of inadmissible traction force parameters, spanned by the basis **Z,** be equal to zero:

$$\left(\mathbf{G}+\mathbf{C}\,\mathbf{R}^{\mathrm{T}}\right)\mathbf{Z} = \mathbf{0} \tag{20}$$

Making use of eq. (12), one obtains the expression of the constants **C**:

$$\mathbf{C} = -\mathbf{G}\,\mathbf{Z}\,\boldsymbol{\lambda}^{-\mathrm{T}} \tag{21}$$

Substitution of **C** into eq. (4), according to its expression above, yields the same eq. (16).

2.2 A Spectrally Consistent Stiffness-Type Matrix

One may define a vector $\mathbf{p}$ of nodal forces that are equivalent in terms of virtual work to the traction force parameters $\mathbf{t}$ on the boundary

$$\mathbf{p} = \mathbf{L}\mathbf{t}\,, \text{ in which } \mathbf{L} \equiv L_{mn} = \int_\Gamma u_{im} t_{in} d\Gamma \tag{22}$$

Then, it follows from eqs. (18) and (22) that

$$\mathbf{p} = \mathbf{K}_C \mathbf{d} - \mathbf{L}\mathbf{G}_a^{(-1)} \mathbf{b}_a\,, \text{ in which } \mathbf{K}_C \equiv \mathbf{L}\mathbf{G}_a^{(-1)}\mathbf{H} \tag{23}$$

is a stiffness-type matrix obtained in the frame of the conventional boundary element method. There is no reason to believe that this matrix should be symmetric, or at least less non-symmetric, in general, than the stiffness-type matrix $\mathbf{L}\mathbf{G}^{-1}\mathbf{H}$. The criticisms expressed by Dumont (1987) are still valid in case of an admissible matrix $\mathbf{G}_a$. However, the matrix $\mathbf{K}_C$, as given in eq.(23), has improved spectral properties that ensure the equilibrium of the equivalent nodal forces $\mathbf{p}$. This shall be demonstrated in the following.

Let the columns of a rectangular matrix $\mathbf{W} \equiv W_{ns}$ be a basis of the nodal displacements $\mathbf{d}$ related to rigid body displacements. For the moment, one can only say that $\mathbf{W}$ and $\mathbf{Z}$ have the same dimension. For a finite domain, it follows from eq. (5) that, necessarily,

$$\mathbf{H}\mathbf{W} = \mathbf{0} \tag{24}$$

which is a feature related to the physical nature of the fundamental solution. On the other hand, the rigid body displacement functions u_{is}^r may be described along the boundary Γ as a linear combination of the displacement interpolation functions u_{in} and W_{ns}:

$$u_{is}^r = u_{im} W_{mt} \omega_{ts} \tag{25}$$

in which $\boldsymbol{\omega} \equiv \omega_{ts}$ is a non-singular square matrix that transforms W_{mt} into the nodal displacements related to u_{is}^r. Then, it follows from eqs. (11), (22) and (25) that

$$\mathbf{R} = \mathbf{L}^\mathrm{T} \mathbf{W} \boldsymbol{\omega} \tag{26}$$

and, according to eq. (12),

$$\mathbf{L}^\mathrm{T} \mathbf{W} = \mathbf{Z} \boldsymbol{\lambda} \boldsymbol{\omega}^{-1} \tag{27}$$

that is, the columns of $\mathbf{L}^T\mathbf{W}$ lie in the space spanned by the rows of $\mathbf{Z}$. Then

$$\mathbf{W}^\mathrm{T}\mathbf{L}(\mathbf{I} - \mathbf{Z}\mathbf{Z}^\mathrm{T}) = \boldsymbol{\omega}^{-\mathrm{T}} \boldsymbol{\lambda}^\mathrm{T} \mathbf{Z}^\mathrm{T} (\mathbf{I} - \mathbf{Z}\mathbf{Z}^\mathrm{T}) \equiv \mathbf{0} \tag{28}$$

Then, given the definitions of $\mathbf{G}_a^{(-1)}$ in eq. (19) and $\mathbf{K}_C$ in eq. (23), one obtains from the orthogonality conditions expressed in eqs. (24) and (28) that $\mathbf{W}^\mathrm{T}\mathbf{K}_C = \mathbf{K}_C \mathbf{W}^\mathrm{T} = \mathbf{0}$. As a consequence, the equivalent nodal forces $\mathbf{p}$ of eq. (23) are always self-equilibrated. Moreover, it may be demonstrated that $\mathrm{rank}(\mathbf{K}_C) = \mathrm{rank}\left(\mathbf{I} - \mathbf{W}\mathbf{W}^\mathrm{T}\right)$.

3. THE HYBRID DISPLACEMENT BOUNDARY ELEMENT METHOD

The hybrid displacement boundary element method was introduced by De Figueiredo (1991), as an alternative to the hybrid stress boundary element method (Dumont, 1987a).

When the rigid body displacements of the fundamental solution are properly considered, the variational principle that underlies the hybrid displacement boundary element method becomes, in matrix notation,

$$\delta\mathbf{p}^{*\mathbf{T}}\left[-\mathbf{F}\mathbf{p}^{*}+\mathbf{b}+\mathbf{C}\left(\mathbf{R}^{\mathbf{T}}\mathbf{t}+\mathbf{b}^{r}\right)+\mathbf{G}\mathbf{t}\right]+\delta\mathbf{d}^{\mathbf{T}}\left[\mathbf{p}-\mathbf{L}\mathbf{t}\right]+ \\ +\delta\mathbf{t}^{\mathbf{T}}\left[\mathbf{G}^{\mathbf{T}}\mathbf{p}^{*}-\mathbf{L}^{\mathbf{T}}\mathbf{d}+\mathbf{R}\mathbf{C}^{\mathbf{T}}\mathbf{p}^{*}\right]=\mathbf{0} \tag{29}$$

in which all quantities have already been defined, with exception of the flexibility matrix

$$\mathbf{F}\equiv F_{mn}=\int_{\Gamma}p_{im}^{*}u_{in}^{*}d\Gamma-\int_{\Omega}\sigma_{ijm}^{*},_{j}u_{in}^{*}d\Omega \tag{30}$$

A vector $\mathbf{p}^{*}$ of singular force parameters is introduced in eq. (29). Note that the rigid body displacements that affect the displacements u_{in}^{*}, according to eq. (1), have no influence in the expression of $\mathbf{F}$, since the forces of a fundamental solution are self-equilibrated by definition and perform zero work on rigid body displacements. This matrix is by definition symmetric. Its integral expression involves the same kind of singularities of the integrals required in the evaluation of the matrices **H** and **G**, except for the elements about the main diagonal, when indices *m* and *n* refer to the same nodal point (Dumont, 1987a). These elements can only be evaluated in the frame of a spectral property to be obtained presently.

For arbitrary variations $\delta\mathbf{p}^{*}$ and $\delta\mathbf{d}$, the set of equations originated from the variational principle may be expressed as

$$\mathbf{F}\mathbf{p}^{*}=\mathbf{G}_{a}\mathbf{t}+\mathbf{b}_{a} \tag{31}$$

$$\mathbf{p}=\mathbf{L}\mathbf{t} \tag{32}$$

$$\mathbf{L}^{\mathbf{T}}\mathbf{d}=\mathbf{G}_{a}^{\mathbf{T}}\mathbf{p}^{*} \tag{33}$$

in which $\mathbf{G}_{a}\equiv\mathbf{G}+\mathbf{C}\mathbf{R}^{\mathbf{T}}\equiv\mathbf{G}\left(\mathbf{I}-\mathbf{Z}\mathbf{Z}^{\mathbf{T}}\right)$ and $\mathbf{b}_{a}\equiv\mathbf{b}+\mathbf{C}\mathbf{b}^{r}\equiv\mathbf{b}-\mathbf{G}\mathbf{Z}\lambda^{-\mathbf{T}}\mathbf{b}^{r}$ are defined according to eqs. (16) and (21).

Making use of the knowledge gained in the first part of this paper, one solves the first equation of the set above:

$$\mathbf{t}=\mathbf{G}_{a}^{(-1)}\mathbf{F}\mathbf{p}^{*}-\mathbf{G}_{a}^{(-1)}\mathbf{b}_{a} \tag{34}$$

in which $\mathbf{G}_{a}^{(-1)}$ is given by eq. (19). Equation (34) means that

$$\mathbf{Z}^{\mathbf{T}}\mathbf{t}=\mathbf{0} \tag{35}$$

and as a consequence, eq. (32) may be written as

$$\mathbf{p}=\mathbf{L}\left(\mathbf{I}-\mathbf{Z}\mathbf{Z}^{\mathbf{T}}\right)\mathbf{t} \tag{36}$$

from which follows that

$$\mathbf{W}^{\mathbf{T}}\mathbf{p}=\mathbf{W}^{\mathbf{T}}\mathbf{L}\left(\mathbf{I}-\mathbf{Z}\mathbf{Z}^{\mathbf{T}}\right)\mathbf{t}=\mathbf{0} \tag{37}$$

according to eq. (28). This demonstrates the spectral consistency of eqs. (32) and (34).

The admissible matrix $\mathbf{G}_{a}$ is by construction orthogonal to **Z**, according to eq. (16). As a consequence, there also exists an orthonormal basis **Y** such that

$$\mathbf{G}_{a}^{\mathbf{T}}\mathbf{Y}=\mathbf{0} \tag{38}$$

Then, the admissible set of singular forces $\mathbf{p}^{*}$, which may be transformed into displacements in eq. (33), must necessarily be orthogonal to **Y**:

$$\mathbf{Y}^{\mathbf{T}}\mathbf{p}^{*}=\mathbf{0} \tag{39}$$

As a consequence, it the matrix **F** in eqs. (31) and (34) is singular, as proposed by Dumont

(1987a) and followed by De Figueiredo (1991), it also must be orthogonal to **Y**:

$$\mathbf{FY} = \mathbf{0} \tag{40}$$

This is the criterion needed for the determination of the elements about the main diagonal of the matrix **F**. An exhaustive investigation of the properties of matrices **F** and $\mathbf{G}_a$ deserves a more extensive paper. One summarizes that, since eqs. (27) and (28) hold, $\mathbf{p}^*$ in eq. (33) may be expressed as

$$\mathbf{p}^* = \left(\mathbf{LG}_a^{(-1)}\right)^{\mathbf{T}} \mathbf{d} \tag{41}$$

Substituting for $\mathbf{p}^*$ in eq. (31) and considering eqs. (32) and (40), one arrives at a stiffness relation between nodal displacements and equivalent nodal forces

$$\mathbf{K}_D \mathbf{d} + \left(\mathbf{LG}_a^{(-1)}\right)\mathbf{b}_a = \mathbf{p}, \text{ in which } \mathbf{K}_D = \left(\mathbf{LG}_a^{(-1)}\right)\mathbf{F}\left(\mathbf{LG}_a^{(-1)}\right)^{\mathbf{T}} \tag{42}$$

is a stiffness matrix. According to eq. (28), $\mathbf{K}_D$ is by construction orthogonal to rigid body displacements, independently from the properties of the matrix $\mathbf{F}$.

De Figueiredo introduced the hybrid displacement boundary element method with no consideration of the rigid body displacements that are inherent to a fundamental solution. Then, she had **C = 0** in eq. (29) and the matrix **G** in place of $\mathbf{G}_a$ in eqs. (31) and (33). Moreover, it was assumed that, instead of eq. (40),

$$\mathbf{F}\tilde{\mathbf{Y}} = \mathbf{0} \tag{43}$$

where $\tilde{\mathbf{Y}}$ is the solution of eq. (33) for inadmissible displacements (with **G** in place of $\mathbf{G}_a$):

$$\mathbf{G}^{\mathbf{T}}\tilde{\mathbf{Y}} = \mathbf{L}^{\mathbf{T}}\mathbf{W} \tag{44}$$

thus arriving, after evaluation of the diagonal elements of **F**, according to eq. (43), at

$$\mathbf{K}_D \mathbf{d} + \left(\mathbf{LG}^{-1}\right)\mathbf{b} = \mathbf{p} \quad \text{with} \quad \mathbf{K}_D = \left(\mathbf{LG}^{-1}\right)\mathbf{F}\left(\mathbf{LG}^{-1}\right)^{\mathbf{T}} \tag{45}$$

To assess the coherence of this formulation, consider eq. (38) written as

$$\left(\mathbf{I} - \mathbf{ZZ}^{\mathbf{T}}\right)\mathbf{G}^{\mathbf{T}}\mathbf{Y} = \mathbf{0} \tag{46}$$

Then, it follows that

$$\mathbf{G}^{\mathbf{T}}\mathbf{Y} = \mathbf{ZZ}^{\mathbf{T}}\mathbf{G}^{\mathbf{T}}\mathbf{Y} \equiv \tilde{\mathbf{Z}} \tag{47}$$

in which $\tilde{\mathbf{Z}}$ is a non-orthonormal basis of the same space spanned by **Z**. Now, observing eqs. (44) and (47), and considering eq. (27), one concludes that $\tilde{\mathbf{Y}}$ in eqs. (43) and (44) is a non-orthonormal basis of the space spanned by **Y**. As a consequence, eqs. (40) and (43) are equivalent. Moreover, matrix $\mathbf{K}_D$, as indicated in both eqs. (42) and (45), is one and the same matrix, provided that **G** may be inverted (is not ill conditioned).

The brief outline of this section is an important theoretical contribution to the hybrid displacement boundary element method, since it assesses ant attests the spectral consistency of the stiffness matrix $\mathbf{K}_D$, obtained by De Figueiredo (1991). However, the vector of nodal forces equivalent to body forces should be expressed as in eq. (42), not as in eq. (45), for the complete consistency of the formulation.

4. AN OUTLINE OF THE HYBRID STRESS BOUNDARY ELEMENT METHOD

The hybrid stress boundary element method is based on the Hellinger-Reissner potential. It was first applied by Pian to finite elements. In 1987, Dumont generalized Pian's ideas for considering the stress field in the domain as a series of fundamental, singular solutions, thus arriving at a boundary integral formulation. In matrix notation, the variational principle writes

$$-\delta\Pi_R = \delta\mathbf{p}^{*\mathbf{T}}\left[\mathbf{F}\mathbf{p}^* + \mathbf{b}^b - \mathbf{H}\mathbf{d}\right] + \delta\mathbf{d}^{\mathbf{T}}\left[\mathbf{p} - \mathbf{t}^b - \mathbf{H}^{\mathbf{T}}\mathbf{p}^*\right] = \mathbf{0} \tag{48}$$

The flexibility matrix **F** and the cinematic transformation matrix **H** have already been defined, although in a different context, in eqs. (30) and (6), respectively. The vector **p** is in part a set of nodal forces equivalent to known surface forces t_i along part Γ_σ of the boundary:

$$\mathbf{p} \equiv p_m = \int_{\Gamma_\sigma} u_{im} t_i d\Gamma \tag{49}$$

and in part a set of unknowns corresponding to reaction forces along the complementary boundary segment Γ_u. The vectors $\mathbf{b}^b$ and $\mathbf{t}^b$ are nodal displacements and nodal forces, respectively, equivalent to applied body forces:

$$\mathbf{b}^b \equiv b_m^b = \int_\Gamma p_{im}^* u_{in}^b d\Gamma - \int_\Omega \sigma_{ijm}^*,_j\, u_{in}^b d\Omega, \qquad \mathbf{t}^b \equiv t_m^b = \int_\Gamma \sigma_{ij}^b \eta_j u_{im}^b d\Gamma \tag{50}$$

in which σ_{ij}^b is an arbitrary stress field in equilibrium with the applied body forces, such that $\sigma_{ij}^b,_j + b_i = 0$ in Ω, and u_{in}^b are the corresponding displacements. Note that neither eq. (30) nor eq. (50) is affected by the rigid body displacements inherent to u_{in}^* and u_{in}^b, respectively, since the stresses defined in a fundamental solution are by construction self equilibrated.

For arbitrary variations $\delta\mathbf{p}^*$ and $\delta\mathbf{d}$, two sets of equations originate from eq. (48):

$$\begin{aligned} \mathbf{F}\mathbf{p}^* &= \mathbf{H}\mathbf{d} - \mathbf{b}^b \\ \mathbf{H}^{\mathbf{T}}\mathbf{p}^* &= \mathbf{p} - \mathbf{t}^b \end{aligned} \tag{51}$$

For a finite domain, the matrix **H** is singular by construction, as formalized in eq. (24). As a consequence, there is an orthogonal basis **V** such that

$$\mathbf{H}^{\mathbf{T}}\mathbf{V} = \mathbf{0} \tag{52}$$

Moreover, it may be verified that, in the second of eqs. (51),

$$\mathbf{W}^{\mathbf{T}}\left(\mathbf{p} - \mathbf{t}^b\right) = \mathbf{0} \tag{53}$$

Then, one must have, for physical consistency,

$$\mathbf{V}^{\mathbf{T}}\mathbf{p}^* = \mathbf{0} \tag{54}$$

from which follows, in the first of eqs. (51), that necessarily

$$\mathbf{F}\mathbf{V} = \mathbf{0} \tag{55}$$

This equation is the key for the evaluation of the elements about the main diagonal of the matrix **F**, which cannot be directly obtained by integration.

Considering the spectral properties given by eqs. (54) and (55), one may solve the first of eqs. (51) for $\mathbf{p}^*$, in terms of generalized inverses (Ben-Israel and Greville, 1980) and introduce its expression into the second of eqs. (51), thus arriving at the relation

$$\mathbf{H}^{\mathrm{T}}\left(\mathbf{F}+\mathbf{V}\mathbf{V}^{\mathrm{T}}\right)^{-1}\mathbf{H}\mathbf{d}=\mathbf{p}-\mathbf{t}^{b}+\mathbf{H}^{\mathrm{T}}\left(\mathbf{F}+\mathbf{V}\mathbf{V}^{\mathrm{T}}\right)^{-1}\mathbf{b}^{b} \quad (56)$$

in which $\mathbf{H}^{\mathrm{T}}\left(\mathbf{F}+\mathbf{V}\mathbf{V}^{\mathrm{T}}\right)^{-1}\mathbf{H}\equiv\mathbf{K}_{S}$ is a symmetric, positive semi-definite stiffness matrix. Owing to the spectral property of **H** given by eq. (24), this stiffness matrix is by construction orthogonal to rigid body displacements.

For the sake of brevity, one has to content oneself with this short description of the hybrid stress boundary element method. Interested readers are referred to some of the articles written by the author in the last decade.

5. A COMPARATIVE SPECTRAL ANALYSIS OF THE METHODS OUTLINED

The three methods presented are schematized in Figs. 1, 2 and 3. One readily identifies all types of transformations performed between the different coordinate systems, taking into account the bases **V**, **Y**, **W** and **Z** of inadmissible parameters. All transformations are physically interpreted. Moreover, all primary nodal parameters are identified in brackets, according to which one can represent the final results both in the domain and along the boundary.

For the sake of brevity, numerical results could not be considered in this article. All three formulations perform equivalently, in terms of both accuracy and spectral properties, provided that one considers the admissible matrix $\mathbf{G}_a$ of eq. (16) and proceeds as outlined. Use of the inconsistent matrix **G** may lead to hazardous results, in case of ill conditioning.

All considerations in this paper were made for a finite, simply connected domain. For either a multiply connected or an infinite domain, some new considerations have to be added, in general, although the basic spectral properties remain valid.

The author hopes to have accomplished his task: demonstrate that in all boundary element formulations one has to deal with singular matrices and generalized inverses. A not unremarkable conclusion is that both the conventional and the hybrid displacement boundary element methods need some conceptual improvements in their formulations, in order to become completely consistent.

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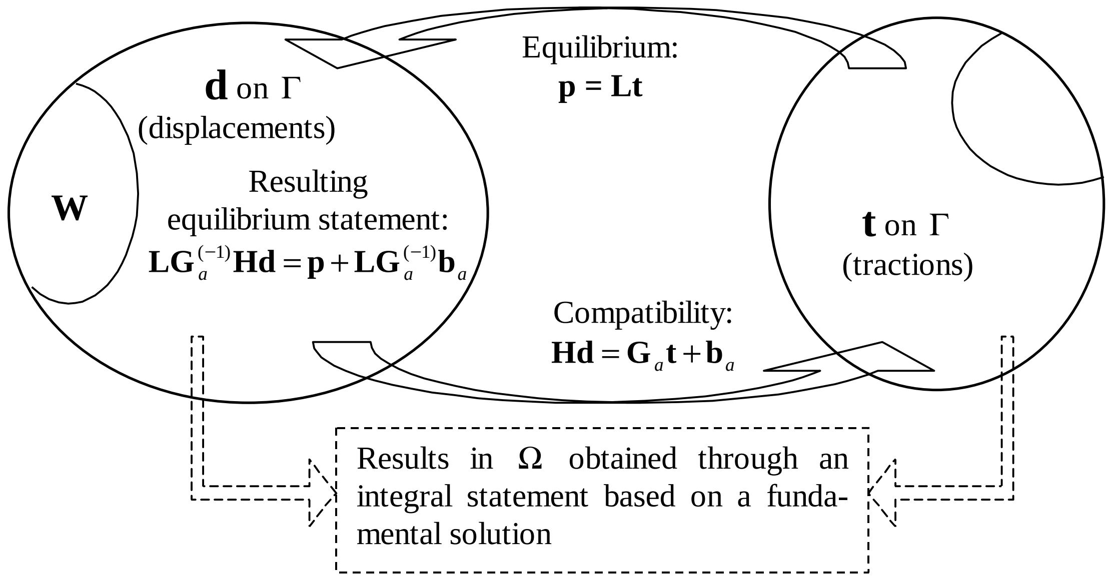

Figure 1. Transformations carried out in the conventional boundary element method.

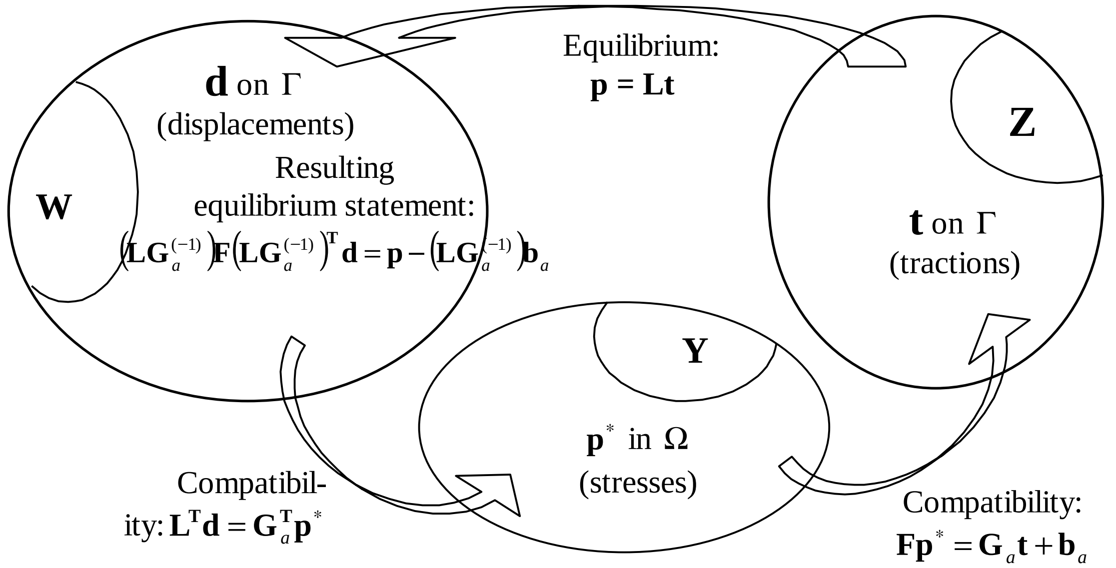

Figure 2. Transformations carried out in the hybrid displacement boundary element method.

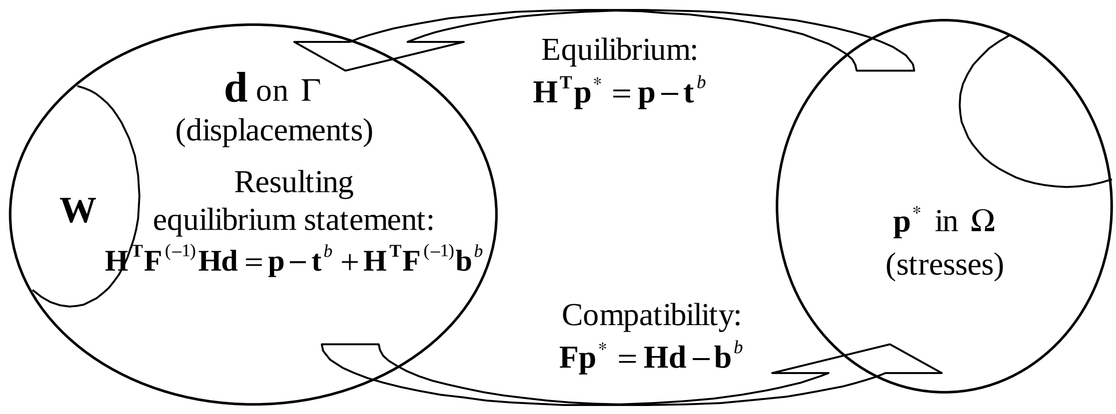

Figure 3. Transformations carried out in the hybrid stress boundary element method.